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二维分数阶强耦合薛定谔方程的保结构方法

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摘要:构造能够保持分数阶强耦合薛定谔方程原始不变量的有效数值解法. 首先利用降阶技术和实部、虚部分离手段将分数阶强耦合薛定谔方程改写成等价的哈密顿系统,然后在空间和时间方向分别采用 Fourier 拟谱法和分区平均向量场(PAVF)系列方法进行离散,建立相应的全离散数值方法. 理论和数值实验结果表明,所获得的 PAVF 系列方法都能够保持模型的原始能量,但只有 PAVF-P 方法能同时保持原始的能量和质量.

关键词:哈密顿系统; 耦合薛定谔方程; 平均向量场方法; Fourier 谱法

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0 引言

和传统的整数阶微积分相比,分数阶微积分凭借其积分定义形式,往往表现出更好的非局部依赖性,能够更好地解释一些具有记忆和遗传特征的模型和现象. 因此,近年来分数阶微积分在物理和工程中得到了大量应用,各种基于分数阶微积分建立的模型被不断提出[1].

文献[2-3]首先利用 Lévy 路径积分,建立了 Riesz 空间分数阶导数情形的分数阶 Schrödinger 方程,并且对分数阶和整数阶方程之间的关系进行了讨论. 文献[4-5]给出了更多的关于分数阶 Schrödinger 方程的一些物理应用. 对于分数阶 Schrödinger 方程(FSEs)和耦合分数阶 Schrödinger 方程(CFSEs)相关的其他属性,例如孤子动力学、基态、吸引子和一致性等[6-10].

本文主要考虑如下强耦合空间分数阶 Schrödinger 方程(SCFSEs)[11-14]

$$\begin{gathered}\mathrm{i}u_t+\gamma_1(-\Delta)^{\frac{\alpha}{2}}u+(\gamma_2|u|^2+\\ \gamma_3|v|^2)u+\gamma_4u+\gamma_5v=0,\\ \boldsymbol{x}\in\Omega,\quad t\in(0,T],\end{gathered}\tag{1}$$

$$\begin{gathered}\mathrm{i}v_t+\gamma_1(-\Delta)^{\frac{\alpha}{2}}v+(\gamma_2|v|^2+\gamma_3|u|^2)v+\\ \gamma_4v+\gamma_5u=0,\\ \boldsymbol{x}\in\Omega,\quad t\in(0,T].\end{gathered}\tag{2}$$

初始条件为

$$\begin{gathered}u(\boldsymbol{x},0)=u_0(\boldsymbol{x}),\\ v(\boldsymbol{x},0)=v_0(\boldsymbol{x}),\quad \boldsymbol{x}\in\Omega,\end{gathered}\tag{3}$$

其中, $\mathrm{i}=\sqrt{-1}$, $1<\alpha\leqslant 2$, $\gamma_1>0$ 是群速度色散, γ_2 描述了双折射介质中脉冲信号的自聚焦, γ_3 是定义方程(1)和(2)可积性的交叉相位调制(交互相位调变), γ_4 是归一化双折射常数,表现为恒定环境电势, γ_5 被称为线性耦合参数,也被称为线性双折射. $u_0(\boldsymbol{x})$ 和 $v_0(\boldsymbol{x})$ 是已知的光滑复值函数,分数阶 Laplacian 算子 $(-\Delta)^{\frac{\alpha}{2}}$ 被定义为[15]

$$(-\Delta)^{\frac{\alpha}{2}}u(x,t)=\mathscr{F}^{-1}[|\xi|^{\alpha}\mathscr{F}(u(\xi,t))],\tag{4}$$

其中, $\mathscr{F}$ 是 Fourier 变换, $\mathscr{F}^{-1}$ 为 Fourier 逆变换. 该模型被广泛应用于量子物理学、非线性光学、等离子体物理学和流体力学等领域.

注意到, SCFSEs(1)和(2)有如下守恒特征:

$$\begin{gathered}Q(t)=Q(0),\quad E(t)=E(0),\\ \forall t>0,\end{gathered}\tag{5}$$

其中质量

$$Q(t)=\|u(\cdot,t)\|_{L^2}^2+\|v(\cdot,t)\|_{L^2}^2,\tag{6}$$

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能量

$$E(t)=\gamma_1(\|(-\Delta)^{\frac{\alpha}{4}}u(\cdot,t)\|_{L^2}^2+\|(-\Delta)^{\frac{\alpha}{4}}v(\cdot,t)\|_{L^2}^2)-\frac{1}{2}\gamma_2(\|u(\cdot,t)\|_{L^4}^4+\|v(\cdot,t)\|_{L^4}^4)-\gamma_3\int_\Omega|u(\cdot,t)|^2|v(\cdot,t)|^2\mathrm{d}\boldsymbol{x}-\gamma_4(\|u(\cdot,t)\|_{L^2}^2+\|v(\cdot,t)\|_{L^2}^2)-2\gamma_5\mathrm{Re}(\int_\Omega u\bar{v}\mathrm{d}\boldsymbol{x}).\quad(7)$$

随着研究的不断深入,包括工程师在内的科技人员对数值算法也提出了更高的要求.尤其是针对这类拥有重要特征的物理模型,他们希望相应的数值方法在离散的数值层面也能反映出相应的物理特征.事实上,“……在某些领域,能否保持原微分方程的某些不变性质是判断一个数值模拟是否成功的标准”[16].因此,构造保持SCFSEs(1)和(2)守恒性质(5)的有效数值解法具有重要的理论和应用价值.

到目前为止,已有不少学者提出了一些求解FSEs的有效数值方法.Wang等[17]针对单个的FSEs构造了相应的能量守恒差分格式;Wang等[18]针对CFSEs提出了相应的线性化隐式守恒差分格式,并进行了详细的误差分析;Ran等[12]进一步针对SCFSEs提出了相应隐式和线性化守恒差分方法,并进行了细致的数值验证;随后,一些研究者也从Galerkin有限元和谱方法的角度构造了相应的守恒方法[19-21].

然而上述研究还主要集中于一维问题,且在针对CFSEs的研究中,现有的绝大部分方法都只能保持原方程的修正能量或质量,而非原始能量与质量.受文献[22-23]的启发,本文拟从模型的哈密顿结构角度,联合PAVF方法去构建数值求解多维SCFSEs的守恒方法.同时希望所构造的数值方法可以在离散意义下既保持原始质量也保持原始能量.

1 SCFSEs(1)和(2)的哈密顿结构

鉴于直接对系统(1)和(2)中的微分算子进行差商逼近所构造的数值方法存在耗时长或只保持修正物理量的不足,以及拥有哈密顿结构的系统在构造其保结构数值方法时相对比较容易的事实,本节旨在探讨耦合系统(1)~(2)的哈密顿形式.

在这之前,首先引进如下重要引理.

引理2.1[24] 对给定$\alpha>0$和任意实值周期函数$p,q\in L_\Omega^2$,有

$$\int_\Omega(-\Delta)^{\frac{\alpha}{2}}pq\mathrm{d}\boldsymbol{x}=\int_\Omega(-\Delta)^{\frac{\alpha}{4}}p(-\Delta)^{\frac{\alpha}{4}}q\mathrm{d}\boldsymbol{x}=\int_\Omega p(-\Delta)^{\frac{\alpha}{2}}q\mathrm{d}\boldsymbol{x}.\quad(8)$$

引理2.2[25] 设泛函

$$\mathscr{F}[g]=\int_\Omega f(g(\boldsymbol{x}),(-\Delta)^{\frac{\alpha}{4}}g(\boldsymbol{x}))\mathrm{d}\boldsymbol{x},\quad(9)$$

其中f是定义在Ω上的光滑函数,则有变分导数

$$\frac{\delta\mathscr{F}}{\delta g}=\frac{\partial f}{\partial g}+(-\Delta)^{\frac{\alpha}{4}}\frac{\partial f}{\partial((-\Delta)^{\frac{\alpha}{4}}g)}.\quad(10)$$

令$u=p+\mathrm{i}q,v=\varphi+\mathrm{i}\psi$,可将式(1)和(2)改写,再分离改写式中的实部和虚部,可进一步得到如下一阶耦合系统

$$p_t=\gamma_1(-\Delta)^{\frac{\alpha}{2}}q-(\gamma_2p^2+\gamma_2q^2+\gamma_3\varphi^2+\gamma_3\psi^2)q-\gamma_4q-\gamma_5\psi,\quad(11)$$

$$q_t=-\gamma_1(-\Delta)^{\frac{\alpha}{2}}p+(\gamma_2p^2+\gamma_2q^2+\gamma_3\varphi^2+\gamma_3\psi^2)p+\gamma_4p+\gamma_5\varphi,\quad(12)$$

$$\varphi_t=\gamma_1(-\Delta)^{\frac{\alpha}{2}}\psi-(\gamma_2\varphi^2+\gamma_2\psi^2+\gamma_3p^2+\gamma_3q^2)\psi-\gamma_4\psi-\gamma_5q,\quad(13)$$

$$\psi_t=-\gamma_1(-\Delta)^{\frac{\alpha}{2}}\varphi+(\gamma_2\varphi^2+\gamma_2\psi^2+\gamma_3p^2+\gamma_3q^2)\varphi+\gamma_4\varphi+\gamma_5p.\quad(14)$$

可以证明耦合系统(11)~(14)满足如下守恒定理.

定理2.1 在周期性边界条件下,耦合系统(11)~(14)满足守恒特征

$$\frac{\mathrm{d}G}{\mathrm{d}t}=0,\quad\frac{\mathrm{d}H}{\mathrm{d}t}=0,\quad(15)$$

其中质量

$$G=\int_\Omega(p^2+q^2+\varphi^2+\psi^2)\mathrm{d}\boldsymbol{x},\quad(16)$$

能量

$$H=\int_\Omega(\gamma_1(((-\Delta)^{\frac{\alpha}{4}}p)^2+((-\Delta)^{\frac{\alpha}{4}}q)^2+((-\Delta)^{\frac{\alpha}{4}}\varphi)^2+((-\Delta)^{\frac{\alpha}{4}}\psi)^2)-$$

$$\frac{\gamma_2}{2}((p^2+q^2)^2+(\varphi^2+\psi^2)^2)-\gamma_3((p^2+q^2)(\varphi^2+\psi^2))-\gamma_4(p^2+q^2+\varphi^2+\psi^2)-2\gamma_5(p\varphi+q\psi))\mathrm{d}\boldsymbol{x}. \tag{17}$$

证明　将式(11)~(14)分别与 p、q、φ、ψ 作内积,并将所得各式相加,再利用引理 2.1,即可得到式(15).

同理,将式(11)~(14)分别与 p_t, $-q_t$, φ_t, $-\psi_t$ 作内积,并将所得各式相加即可得到式(15). 证毕.

定理 2.2　耦合系统式(11)~(14)可改写为如下无穷维哈密顿系统

$$\begin{bmatrix} p_t \\ q_t \\ \varphi_t \\ \psi_t \end{bmatrix} = J\begin{bmatrix} \frac{\delta H}{\delta p} \\ \frac{\delta H}{\delta q} \\ \frac{\delta H}{\delta \varphi} \\ \frac{\delta H}{\delta \psi} \end{bmatrix},$$

其中

$$J=\begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & 0 \end{bmatrix} \tag{18}$$

是一个反对称矩阵.

证明　利用引理 2.2,可得

$$\frac{\delta H}{\delta p}=2(\gamma_1(-\Delta)^{\frac{\alpha}{2}}p-(\gamma_2 p^2+\gamma_2 q^2+\gamma_3\varphi^2+\gamma_3\psi^2)p-\gamma_4 p-\gamma_5\varphi), \tag{19}$$

$$\frac{\delta H}{\delta q}=2(\gamma_1(-\Delta)^{\frac{\alpha}{2}}q-(\gamma_2 p^2+\gamma_2 q^2+\gamma_3\varphi^2+\gamma_3\psi^2)q-\gamma_4 q-\gamma_5\psi), \tag{20}$$

$$\frac{\delta H}{\delta \varphi}=2(\gamma_1(-\Delta)^{\frac{\alpha}{2}}\varphi-(\gamma_2\varphi^2+\gamma_2\psi^2+\gamma_3 p^2+\gamma_3 q^2)\varphi-\gamma_4\varphi-\gamma_5 p), \tag{21}$$

$$\frac{\delta H}{\delta \psi}=2(\gamma_1(-\Delta)^{\frac{\alpha}{2}}\psi-(\gamma_2\varphi^2+\gamma_2\psi^2+\gamma_3 p^2+\gamma_3 q^2)\psi-\gamma_4\psi-\gamma_5 q). \tag{22}$$

注意到耦合系统(11)~(14),有

$$\begin{bmatrix} p_t \\ q_t \\ \varphi_t \\ \psi_t \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & 0 \end{bmatrix}\begin{bmatrix} \frac{\delta H}{\delta p} \\ \frac{\delta H}{\delta q} \\ \frac{\delta H}{\delta \varphi} \\ \frac{\delta H}{\delta \psi} \end{bmatrix}. \tag{23}$$

证毕.

2　SCFSEs(1)和(2)的守恒格式

2.1　符号　为了简单,本文考虑二维矩形区域 $\Omega=(-L,L)\times(-L,L)$. 记 $t_n=n\tau$, $x_i=-L+ih$, $y_j=-L+jh$,其中 τ、h 分别表示时间和空间方向的网格步长. 记 $x_{ij}=(x_i,y_j)$, $0\leqslant n\leqslant N$, $0\leqslant i,j\leqslant M$,其中 $\tau=T/N$, $h=2L/M$, M 和 N 是 2 个正整数. 定义离散网格 $\Omega_h^\tau=\Omega_h\times\Omega_\tau$,其中 $\Omega_\tau=\{t_n\mid t_n=n\tau, 0\leqslant n\leqslant N\}$, $\Omega_h=\{(x_i,y_j)\mid i,j=0,1,\cdots,M-1\}$,边界 $\partial\Omega_h=\{(x_i,y_j)\mid i,j=0 \text{ 或 } M\}$,且 $\bar{\Omega}_h=\Omega_h\cup\partial\Omega_h$.

令 $(p_{ij}^n,q_{ij}^n,\varphi_{ij}^n,\psi_{ij}^n)$ 为精确解 (p,q,φ,ψ) 在网格点 (x_{ij},t_n) 处的数值近似,并记

$$\delta_t P^n=\frac{P^{n+1}-P^n}{\tau},\quad P^{n+\frac{1}{2}}=\frac{P^{n+1}+P^n}{2},$$

其中向量

$$P=(p_{0,0},\cdots,p_{M-1,0},p_{0,1},\cdots,p_{M-1,1},\cdots,p_{0,M-1},\cdots,p_{M-1,M-1})^{\mathrm{T}}.$$

向量 Q、Φ 和 Ψ 可类似定义. 此外,定义离散内积

$$(P,Q)=h^2\sum_{i=1}^{M-1}\sum_{j=1}^{M-1}P_{ij}Q_{ij}$$

和范数

$$\|P\|^2=(P,P),\quad \|P\|_\infty=\sup_{x_{ij}\in\Omega_h}|P_{ij}|.$$

2.2　空间半离散系统　为了使所构造的数值方法具有高阶精度,同时考虑到周期边界条件,采用 Fourier 拟谱方法进行空间离散是一种较好的选择.

设 $(x_i,y_j)\in\Omega_h$ 是 Fourier 配置点,则 $u(x,y)$ 可用多项式

$$u_M(x,y)=\sum_{k_1=-M/2}^{M/2}\sum_{k_2=-M/2}^{M/2}\hat{u}_{k_1,k_2}\mathrm{e}^{\mathrm{i}\mu(k_1(x+L)+k_2(y+L))} \tag{24}$$

进行逼近,其中 $\mu=\pi/L$,系数

$$\hat{u}_{k_1,k_2}=\frac{1}{Mc_{k_1}}\frac{1}{Mc_{k_2}}\sum_{l_1=0}^{M-1}\sum_{l_2=0}^{M-1}u(x_{l_1},y_{l_2})\times \mathrm{e}^{-\mathrm{i}\mu(k_1(x+L)+k_2(y+L))},\tag{25}$$

$$c_{k_l}=\begin{cases}1, & |k_l|<M/2,\\ 2, & |k_l|=M/2,\end{cases}\quad l=1,2.\tag{26}$$

因此,分数阶算子

$$(-\Delta)^{\frac{\alpha}{2}}u_M(x,y)=\sum_{k_1=-M/2}^{M/2}\sum_{k_2=-M/2}^{M/2}|(k_1\mu)^2+(k_2\mu)^2|^{\frac{\alpha}{2}}\times \hat{u}_{k_1,k_2}\mathrm{e}^{\mathrm{i}\mu(k_1(x+L)+k_2(y+L))}.\tag{27}$$

将$\hat{u}_{k_1,k_2}$代入式(27),可得

$$(-\Delta)^{\frac{\alpha}{2}}u_M(x_i,y_j)=\sum_{k_1=-M/2}^{M/2}\sum_{k_2=-M/2}^{M/2}|(k_1\mu)^2+(k_2\mu)^2|^{\frac{\alpha}{2}}\times \left(\frac{1}{Mc_{k_1}}\frac{1}{Mc_{k_2}}\sum_{l_1=0}^{M-1}\sum_{l_2=0}^{M-1}u(x_{l_1},y_{l_2})\mathrm{e}^{-\mathrm{i}\mu(k_1(x+L)+k_2(y+L))}\right)\times \mathrm{e}^{\mathrm{i}\mu(k_1(x_i+L)+k_2(y_j+L))}=\sum_{l_1=0}^{M-1}\sum_{l_2=0}^{M-1}u(x_{l_1},y_{l_2})\times \left(\sum_{k_1=-M/2}^{M/2}\sum_{k_2=-M/2}^{M/2}\frac{1}{Mc_{k_1}}\frac{1}{Mc_{k_2}}\times |(k_1\mu)^2+(k_2\mu)^2|^{\frac{\alpha}{2}}\right)\mathrm{e}^{\mathrm{i}\mu(k_1(x_i-x_{l_1})+k_2(y_j-y_{l_2}))}=(D^\alpha U)_{i+jM},\tag{28}$$

其中D^α是微分对称矩阵,其元素

$$(D^\alpha U)_{i+jM,l_1+l_2M}=\sum_{k_1=-M/2}^{M/2}\sum_{k_2=-M/2}^{M/2}\frac{1}{Mc_{k_1}}\frac{1}{Mc_{k_2}}\times |(k_1\mu)^2+(k_2\mu)^2|^{\frac{\alpha}{2}}\times \mathrm{e}^{\mathrm{i}\mu(k_1(x_i-x_{l_1})+k_2(y_j-y_{l_2}))}.\tag{29}$$

利用Fourier拟谱法离散耦合系统(11)~(14)中的空间分数阶导数,可得如下半离散系统

$$P_t=\gamma_1D^\alpha Q-(\gamma_2P^2+\gamma_2Q^2+\gamma_3\Phi^2+\gamma_3\Psi^2)\cdot Q-\gamma_4Q-\gamma_5\Psi,\tag{30}$$

$$Q_t=-\gamma_1D^\alpha P+(\gamma_2P^2+\gamma_2Q^2+\gamma_3\Phi^2+\gamma_3\Psi^2)\cdot P+\gamma_4P+\gamma_5\Phi,\tag{31}$$

$$\Phi_t=\gamma_1D^\alpha\Psi-(\gamma_2\Phi^2+\gamma_2\Psi^2+\gamma_3P^2+\gamma_3Q^2)\cdot\Psi-\gamma_4\Psi-\gamma_5Q,\tag{32}$$

$$\Psi_t=-\gamma_1D^\alpha\Phi+(\gamma_2\Phi^2+\gamma_2\Psi^2+\gamma_3P^2+\gamma_3Q^2)\cdot\Phi+\gamma_4\Phi+\gamma_5P,\tag{33}$$

其中$P^2=P\cdot P$,“$\cdot$”表示向量之间的点乘运算.

令$Z=(P,Q,\Phi,\Psi)^{\mathrm{T}}$,则上述空间半离散系统可以重写为标准的哈密顿形式

$$\frac{\mathrm{d}Z}{\mathrm{d}t}=f(Z)=S\nabla H(Z),\tag{34}$$

其中哈密顿量

$$H(Z)=\gamma_1(P^{\mathrm{T}}D^\alpha P+Q^{\mathrm{T}}D^\alpha Q+\Phi^{\mathrm{T}}D^\alpha\Phi+\Psi^{\mathrm{T}}D^\alpha\Psi)-\frac{1}{2}\gamma_2((P^2+Q^2)^{\mathrm{T}}(P^2+Q^2)+(\Phi^2+\Psi^2)^{\mathrm{T}}(\Phi^2+\Psi^2))-\gamma_3((P^2+Q^2)^{\mathrm{T}}(\Phi^2+\Psi^2))-\gamma_4(P^{\mathrm{T}}P+Q^{\mathrm{T}}Q+\Phi^{\mathrm{T}}\Phi+\Psi^{\mathrm{T}}\Psi)-2\gamma_5(P^{\mathrm{T}}\Phi+Q^{\mathrm{T}}\Psi),\tag{35}$$

系数矩阵

$$S=\begin{bmatrix}0&\frac{1}{2}I&0&0\\-\frac{1}{2}I&0&0&0\\0&0&0&\frac{1}{2}I\\0&0&-\frac{1}{2}I&0\end{bmatrix}\tag{36}$$

是一个反对称矩阵. I是单位矩阵.

若定义质量

$$G(Z)=P^{\mathrm{T}}P+Q^{\mathrm{T}}Q+\Phi^{\mathrm{T}}\Phi+\Psi^{\mathrm{T}}\Psi,\tag{37}$$

则上述半离散系统具有如下性质.

定理 3.1 半离散系统(30)~(33)满足守恒定律

$$\frac{\mathrm{d}H(Z)}{\mathrm{d}t}=0,\quad \frac{\mathrm{d}Q(Z)}{\mathrm{d}t}=0.\tag{38}$$

证明 注意到矩阵S的反对称性,利用式(34)可得

$$\frac{\mathrm{d}H(Z)}{\mathrm{d}t}=\nabla H(Z)^{\mathrm{T}}f(Z)=\nabla H(Z)^{\mathrm{T}}S\nabla H(Z)=0.\tag{39}$$

根据式(37)可得

$$\frac{\mathrm{d}G(Z)}{\mathrm{d}t}=2h^2(P^{\mathrm{T}}P+Q^{\mathrm{T}}Q+\Phi^{\mathrm{T}}\Phi+\Psi^{\mathrm{T}}\Psi)=2h^2(\gamma_1P^{\mathrm{T}}D^\alpha Q-P^{\mathrm{T}}(\gamma_2P^2+\gamma_2Q^2+\gamma_3\Phi^2+\gamma_3\Psi^2)\cdot Q-\gamma_4P^{\mathrm{T}}Q-\gamma_5P^{\mathrm{T}}\Psi-\gamma_1Q^{\mathrm{T}}D^\alpha P+Q^{\mathrm{T}}(\gamma_2P^2+\gamma_2Q^2+\gamma_3\Phi^2+\gamma_3\Psi^2)\cdot P+\gamma_4Q^{\mathrm{T}}P+\gamma_5Q^{\mathrm{T}}\Phi+$$

$$\gamma_1\Phi^{\mathrm T}D^\alpha\Psi-\Phi^{\mathrm T}(\gamma_2\Phi^2+\gamma_2\Psi^2+\gamma_3P^2+\gamma_3Q^2)\cdot\Psi-\gamma_4\Phi^{\mathrm T}\Psi-\gamma_5\Phi^{\mathrm T}Q-\gamma_1\Psi^{\mathrm T}D^\alpha\Phi+\Psi^{\mathrm T}(\gamma_2\Phi^2+\gamma_2\Psi^2+\gamma_3P^2+\gamma_3Q^2)\cdot\Phi+\gamma_4\Psi^{\mathrm T}\Phi+\gamma_5\Psi^{\mathrm T}P)=0.\quad(40)$$

证毕.

2.3　全离散守恒格式　近年来,围绕哈密顿系统已经提出了不少的守恒方法[26-27],但绝大部分方法在长时间模拟中都只是保持原方程的修正能量或者修正质量.

根据文献[22],知道采用 PAVF 方法对哈密顿系统进行离散,其哈密顿量能够自然得到保持,且还可能保持更多的不变量. 基于此,本节选择采用 PAVF 系列方法去离散上述半离散系统(30)~(33),以期获得能够同时保持原始能量和原始质量的守恒方法.

利用 PAVF 方法离散半离散系统(30)~(33),可得求解 SCFSEs 的全离散格式:

$$\delta_tP^n=\frac12\int_0^1H_Q(P^{n+1},\varepsilon Q^{n+1}+(1-\varepsilon)Q^n,\Phi^n,\Psi^n)\mathrm d\varepsilon,\quad(41)$$

$$\delta_tQ^n=-\frac12\int_0^1H_P(\varepsilon P^{n+1}+(1-\varepsilon)^nP^n,Q^n,\Phi^n,\Psi^n)\mathrm d\varepsilon,\quad(42)$$

$$\delta_t\Phi^nn=\frac12\int_0^1H_\Psi(P^{n+1},Q^{n+1},\Phi^{n+1},\varepsilon\Psi^{n+1}+(1-\varepsilon)\Psi^n)\mathrm d\varepsilon,\quad(43)$$

$$\delta_t\Psi^n=-\frac12\int_0^1H_\Phi(P^{n+1},Q^{n+1},\varepsilon\Phi^{n+1}+(1-\varepsilon)\Phi^n,\Psi^n)\mathrm d\varepsilon,\quad(44)$$

整理得

$$\delta_tP^n=\gamma_1D^\alpha Q^{n+\frac12}-\frac14\gamma_2((Q^{n+1})^3+(Q^n)^3+(Q^{n+1})^2\cdot Q^n+Q^{n+1}\cdot(Q^n)^2)-(\gamma_2(P^{n+1})^2+\gamma_3(\Phi^n)^2+\gamma_3(\Psi^n)^2+\gamma_4)\cdot Q^{n+\frac12}-\gamma_5\Psi^n,\quad(45)$$

$$\delta_tQ^n=-\gamma_1D^\alpha P^{n+\frac12}+\frac14\gamma_2((P^{n+1})^3+(P^n)^3+(P^{n+1})^2\cdot P^n+P^{n+1}\cdot(P^n)^2)+(\gamma_2(Q^n)^2+\gamma_3(\Phi^n)^2+\gamma_3(\Psi^n)^2+\gamma_4)\cdot P^{n+\frac12}+\gamma_5\Phi^n,\quad(46)$$

$$\delta_t\Phi^n=\gamma_1D^\alpha\Psi^{n+\frac12}-\frac14\gamma_2((\Psi^{n+1})^3+(\Psi^n)^3+(\Psi^{n+1})^2\cdot\Psi^n+\Psi^{n+1}\cdot(\Psi^n)^2)-(\gamma_3(P^{n+1})^2+\gamma_3(Q^{n+1})^2+\gamma_2(\Phi^{n+1})^2+\gamma_4)\cdot\Psi^{n+\frac12}-\gamma_5Q^{n+1},\quad(47)$$

$$\delta_t\Psi^n=-\gamma_1D^\alpha\Phi^{n+\frac12}+\frac14\gamma_2((\Phi^{n+1})^3+(\Phi^n)^3+(\Phi^{n+1})^2\cdot\Phi^n+\Phi^{n+1}\cdot(\Phi^n)^2)+(\gamma_3(P^{n+1})^2+\gamma_3(Q^{n+1})^2+\gamma_2(\Psi^n)^2+\gamma_4)\cdot\Phi^{n+\frac12}+\gamma_5P^{n+1},\quad(48)$$

考虑到该格式空间方向采用的是 Fourier 拟谱方法离散,下称其为 FPAVF 方法.

类似地,可得求解 SCFSEs 的伴随 FPAVF 方法:

$$\delta_tP^n=\gamma_1D^\alpha Q^{n+\frac12}-\frac14\gamma_2((Q^{n+1})^3+(Q^n)^3+(Q^{n+1})^2\cdot Q^n+Q^{n+1}\cdot(Q^n)^2)-(\gamma_2(P^n)^2+\gamma_3(\Phi^{n+1})^2+\gamma_3(\Psi^{n+1})^2+\gamma_4)\cdot Q^{n+\frac12}-\gamma_5\Psi^{n+1},\quad(49)$$

$$\delta_tQ^n=-\gamma_1D^\alpha P^{n+\frac12}+\frac14\gamma_2((P^{n+1})^3+(P^n)^3+(P^{n+1})^2\cdot P^n+P^{n+1}\cdot(P^n)^2)+(\gamma_2(Q^{n+1})^2+\gamma_3(\Phi^{n+1})^2+\gamma_3(\Psi^{n+1})^2+\gamma_4)\cdot P^{n+\frac12}+\gamma_5\Phi^{n+1},\quad(50)$$

$$\delta_t\Phi^n=\gamma_1D^\alpha\Psi^{n+\frac12}-\frac14\gamma_2((\Psi^{n+1})^3+(\Psi^n)^3+(\Psi^{n+1})^2\cdot\Psi^n+\Psi^{n+1}\cdot(\Psi^n)^2)-(\gamma_3(P^n)^2+\gamma_3(Q^n)^2+\gamma_2(\Phi^n)^2+\gamma_4)\cdot\Psi^{n+\frac12}-\gamma_5Q^n,\quad(51)$$

$$\delta_t\Psi^n=-\gamma_1D^\alpha\Phi^{n+\frac12}+\frac14\gamma_2((\Phi^{n+1})^3+(\Phi^n)^3+(\Phi^{n+1})^2\cdot\Phi^n+\Phi^{n+1}\cdot(\Phi^n)^2)+(\gamma_3(P^n)^2+\gamma_3(Q^n)^2+\gamma_2(\Psi^{n+1})^2+\gamma_4)\cdot\Phi^{n+\frac12}+\gamma_5P^n,\quad(52)$$

结合 FPAVF 方法(45)~(48)与伴随 FPAVF 方法(49)~(52),可得 FPAVF-P 方法:

$$\delta_tP^n=\gamma_1D^\alpha Q^{n+\frac12}-\frac14\gamma_2((Q^{n+1})^3+(Q^n)^3+$$

$$
\begin{aligned}
&(Q^{n+1})^2\cdot Q^n+Q^{n+1}\cdot(Q^n)^2)-\\
&(\gamma_2\frac{(P^{n+1})^2+(P^n)^2}{2}+\\
&\gamma_3\frac{(\Phi^{n+1})^2+(\Phi^n)^2}{2}+\\
&\gamma_3\frac{(\Psi^{n+1})^2+(\Psi^n)^2}{2}+\\
&\gamma_4)\cdot Q^{n+\frac{1}{2}}-\gamma_5\Psi^{n+\frac{1}{2}},
\end{aligned}
\tag{53}
$$

$$
\begin{aligned}
\delta_tQ^n=&-\gamma_1D^\alpha P^{n+\frac{1}{2}}+\frac{1}{4}\gamma_2((P^{n+1})^3+(P^n)^3+\\
&(P^{n+1})^2\cdot P^n+P^{n+1}\cdot(P^n)^2)+\\
&(\gamma_2\frac{(Q^{n+1})^2+(Q^n)^2}{2}+\\
&\gamma_3\frac{(\Phi^{n+1})^2+(\Phi^n)^2}{2}+\\
&\gamma_3\frac{(\Psi^{n+1})^2+(\Psi^n)^2}{2}+\\
&\gamma_4)\cdot P^{n+\frac{1}{2}}+\gamma_5\Phi^{n+\frac{1}{2}},
\end{aligned}
\tag{54}
$$

$$
\begin{aligned}
\delta_t\Phi^n=&\gamma_1D^\alpha\Psi^{n+\frac{1}{2}}-\frac{1}{4}\gamma_2((\Psi^{n+1})^3+(\Psi^n)^3+\\
&(\Psi^{n+1})^2\cdot\Psi^n+\Psi^{n+1}\cdot(\Psi^n)^2)-\\
&(\gamma_3\frac{(P^{n+1})^2+(P^n)^2}{2}+\\
&\gamma_3\frac{(Q^{n+1})^2+(Q^n)^2}{2}+\\
&\gamma_2\frac{(\Phi^{n+1})^2+(\Phi^n)^2}{2}+\\
&\gamma_4)\cdot\Psi^{n+\frac{1}{2}}-\gamma_5Q^{n+\frac{1}{2}},
\end{aligned}
\tag{55}
$$

$$
\begin{aligned}
\delta_t\Psi^n=&-\gamma_1D^\alpha\Phi^{n+\frac{1}{2}}+\frac{1}{4}\gamma_2((\Phi^{n+1})^3+(\Phi^n)^3+\\
&(\Phi^{n+1})^2\cdot\Phi^n+\Phi^{n+1}\cdot(\Phi^n)^2)+\\
&(\gamma_3\frac{(P^{n+1})^2+(P^n)^2}{2}+\\
&\gamma_3\frac{(Q^{n+1})^2+(Q^n)^2}{2}+\\
&\gamma_2\frac{(\Psi^{n+1})^2+(\Psi^n)^2}{2}+\\
&\gamma_4)\cdot\Phi^{n+\frac{1}{2}}+\gamma_5P^{n+\frac{1}{2}}.
\end{aligned}
\tag{56}
$$

为了进行比较,也用标准的 AVF 方法对系统(30)~(33)进行离散,可得

$$\delta_tP^n=\frac{1}{2}\int_0^1H_Q(\varepsilon Z^{n+1}+(1-\varepsilon)Z^n)\mathrm{d}\varepsilon,\tag{57}$$

$$\delta_tQ^n=-\frac{1}{2}\int_0^1H_P(\varepsilon Z^{n+1}+(1-\varepsilon)Z^n)\mathrm{d}\varepsilon,\tag{58}$$

$$\delta_t\Phi^n=\frac{1}{2}\int_0^1H_\Psi(\varepsilon Z^{n+1}+(1-\varepsilon)Z^n)\mathrm{d}\varepsilon,\tag{59}$$

$$\delta_t\Psi^n=-\frac{1}{2}\int_0^1H_\Phi(\varepsilon Z^{n+1}+(1-\varepsilon)Z^n)\mathrm{d}\varepsilon.\tag{60}$$

整理得

$$
\begin{aligned}
\delta_tP^n=&\gamma_1D^\alpha Q^{n+\frac{1}{2}}-\frac{1}{4}\gamma_2((Q^{n+1})^3+(Q^n)^3+\\
&(Q^{n+1})^2\cdot Q^n+Q^{n+1}\cdot(Q^n)^2)-\\
&\frac{\gamma_2}{12}(3(P^{n+1})^2\cdot Q^{n+1}+3(P^n)^2\cdot Q^n+\\
&(P^n)^2\cdot Q^{n+1}+(P^{n+1})^2\cdot Q^n+\\
&4P^{n+1}\cdot P^n\cdot Q^{n+\frac{1}{2}})-\\
&\frac{\gamma_3}{12}(3(\Phi^{n+1})^2\cdot Q^{n+1}+3(\Phi^n)^2\cdot Q^n+\\
&(\Phi^n)^2\cdot Q^{n+1}+(\Phi^{n+1})^2\cdot Q^n+\\
&4\Phi^{n+1}\cdot\Phi^n\cdot Q^{n+\frac{1}{2}})-\\
&\frac{\gamma_3}{12}(3(\Psi^{n+1})^2\cdot Q^{n+1}+3(\Psi^n)^2\cdot Q^n+\\
&(\Psi^n)^2\cdot Q^{n+1}+(\Psi^{n+1})^2\cdot Q^n+\\
&4\Psi^{n+1}\cdot\Psi^n\cdot Q^{n+\frac{1}{2}})-\\
&\gamma_4Q^{n+\frac{1}{2}}-\gamma_5\Psi^{n+\frac{1}{2}},
\end{aligned}
\tag{61}
$$

$$
\begin{aligned}
\delta_tQ^n=&-\gamma_1D^\alpha P^{n+\frac{1}{2}}+\frac{1}{4}\gamma_2((P^{n+1})^3+(P^n)^3+\\
&(P^{n+1})^2\cdot P^n+P^{n+1}\cdot(P^n)^2)+\\
&\frac{\gamma_2}{12}(3(Q^{n+1})^2\cdot P^{n+1}+3(Q^n)2\cdot P^n+\\
&(Q^n)^2\cdot P^{n+1}+(Q^{n+1})^2\cdot P^n+\\
&4Q^{n+1}\cdot Q^n\cdot P^{n+\frac{1}{2}})+\\
&\frac{\gamma_3}{12}(3(\Phi^{n+1})^2\cdot P^{n+1}+3(\Phi^n)^2\cdot P^n+\\
&(\Phi^n)^2\cdot P^{n+1}+(\Phi^{n+1})^2\cdot P^n+\\
&4\Phi^{n+1}\cdot\Phi^n\cdot P^{n+\frac{1}{2}})+\\
&\frac{\gamma_3}{12}(3(\Psi^{n+1})^2\cdot P^{n+1}+3(\Psi^n)^2\cdot P^n+\\
&(\Psi^n)^2\cdot P^{n+1}+(\Psi^{n+1})^2\cdot P^n+\\
&4\Psi^{n+1}\cdot\Psi^n\cdot P^{n+\frac{1}{2}})+\gamma_4P^{n+\frac{1}{2}}+\gamma_5\Phi^{n+\frac{1}{2}},
\end{aligned}
\tag{62}
$$

$$
\begin{aligned}
\delta_t\Phi^n=&\gamma_1D^\alpha\Psi^{n+\frac{1}{2}}-\frac{1}{4}\gamma_2((\Psi^{n+1})^3+(\Psi^n)^3+\\
&(\Psi^{n+1})^2\cdot\Psi^n+\Psi^{n+1}\cdot(\Psi^n)^2)-\\
&\frac{\gamma_3}{12}(3(P^{n+1})^2\cdot\Psi^{n+1}+3(P^n)^2\cdot\Psi^n+\\
&(P^n)^2\cdot\Psi^{n+1}+(P^{n+1})^2\cdot\Psi^n+
\end{aligned}
$$

$$4P^{n+1}\cdot P^n\cdot \Psi^{n+\frac{1}{2}}) -$$
$$\frac{\gamma_3}{12}(3(Q^{n+1})^2\cdot\Psi^{n+1}+3(Q^n)^2\cdot\Psi^n+(Q^n)^2\cdot\Psi^{n+1}+(Q^{n+1})^2\cdot\Psi^n+4Q^{n+1}\cdot Q^n\cdot\Psi^{n+\frac{1}{2}})-$$
$$\frac{\gamma_2}{12}(3(\Phi^{n+1})^2\cdot\Psi^{n+1}+3(\Phi^n)^2\cdot\Psi^n+(\Phi^n)^2\cdot\Psi^{n+1}+(\Phi^{n+1})^2\cdot\Psi^n+4\Phi^{n+1}\cdot\Phi^n\cdot\Psi^{n+\frac{1}{2}})-\gamma_4\Psi^{n+\frac{1}{2}}-\gamma_5 Q^{n+\frac{1}{2}},\quad(63)$$

$$\delta_t\Psi^n=-\gamma_1 D^\alpha\Phi^{n+\frac{1}{2}}+\frac{1}{4}\gamma_2((\Phi^{n+1})^3+(\Phi^n)^3+(\Phi^{n+1})^2\cdot\Phi^n+\Phi^{n+1}\cdot(\Phi^n)^2)+$$
$$\frac{\gamma_3}{12}(3(P^{n+1})^2\cdot\Phi^{n+1}+3(P^n)^2\cdot\Phi^n+(P^n)^2\cdot\Phi^{n+1}+(P^{n+1})^2\cdot\Phi^n+4P^{n+1}\cdot P^n\cdot\Phi^{n+\frac{1}{2}})+$$
$$\frac{\gamma_3}{12}(3(Q^{n+1})^2\cdot\Phi^{n+1}+3(Q^n)^2\cdot\Phi^n+(Q^n)^2\cdot\Phi^{n+1}+(Q^{n+1})^2\cdot\Phi^n+4Q^{n+1}\cdot Q^n\cdot\Phi^{n+\frac{1}{2}})+$$
$$\frac{\gamma_2}{12}(3(\Psi^{n+1})^2\cdot\Phi^{n+1}+3(\Psi^n)^2\cdot\Phi^n+(\Psi^n)^2\cdot\Phi^{n+1}+(\Psi^{n+1})^2\cdot\Phi^n+4\Psi^{n+1}\cdot\Psi^n\cdot\Phi^{n+\frac{1}{2}})+\gamma_4\Phi^{n+\frac{1}{2}}+\gamma_5 P^{n+\frac{1}{2}}.\quad(64)$$

针对上述的 AVF 或 PAVF 系列方法,有如下结论.

定理 3.2　上述 AVF 或 PAVF 系列方法都满足能量守恒定律

$$H^{n+1}=H^n,\quad(65)$$

但只有 FPAVF-P 方法(53)~(56)还满足质量守恒律

$$G^{n+1}=G^n,\quad(66)$$

其中质量

$$G^n=(P^n)^{\mathrm T}P^n+(Q^n)^{\mathrm T}(Q^n)+(\Phi^n)^{\mathrm T}\Phi^n+(\Psi^n)^{\mathrm T}\Psi^n,\quad(67)$$

能量

$$H^n=\gamma_1((P^n)^{\mathrm T}D^\alpha P^n+(Q^n)^{\mathrm T}D^\alpha Q^n+(\Phi^n)^{\mathrm T}D^\alpha\Phi^n+(\Psi^n)^{\mathrm T}D^\alpha\Psi^n)-$$
$$\frac{\gamma_2}{2}(((P^n)^2+(Q^n)^2)^{\mathrm T}(P^n)^2+(Q^n)^2)+((\Phi^n)^2+(\Psi^n)^2)^{\mathrm T}((\Phi^n)2+(\Psi^n)^2)-$$
$$\gamma_3(((P^n)^2+(Q^n)^2)^{\mathrm T}((\Phi^n)^2+(\Psi^n)^2))-\gamma_4((P^n)^{\mathrm T}P^n+(Q^n)^{\mathrm T}Q^n+(\Phi^n)^{\mathrm T}\Phi^n+(\Psi^n)^{\mathrm T}\Psi^n)-$$
$$2\gamma_5((P^n)^{\mathrm T}\Phi^n+(Q^n)^{\mathrm T}\Psi^n).\quad(68)$$

证明　用 $Q^{n+1}-Q^n$、$-(P^{n+1}-P^n)$、$\Psi^{n+1}-\Psi^n$、$-(\Phi^{n+1}-\Phi^n)$分别与(53)~(56)式作内积,将所得式子相加,即可得 $H^{n+1}=H^n$. 这意味着 FPAVF-P 方法(53)~(56)满足能量守恒律. 按类似的方法可证明上述其他 AVF(61)~(64)和 PAVF(45)~(48)方法也满足能量守恒定律.

将 $P^{n+1}+P^n$、$Q^{n+1}+Q^n$、$\Phi^{n+1}+\Phi^n$、$\Psi^{n+1}+\Psi^n$ 分别与式(53)~(56)作内积,再将所得式子相加,即可证得 $G^{n+1}=G^n$. 证毕.

由于上述其他 AVF(61)~(64)和 PAVF(45)~(48)方法不具备对称性,因此其不满足质量守恒定律. 证毕.

3　数值实验

本节将借助数值算例来验证前面理论结果的正确性.

为此,首先定义如下误差函数

$$E(\tau)=\|U_M^N-U_M^{2N}\|_\infty,\quad E(h)=\|U_M^N-U_{2M}^N\|_\infty,\quad(69)$$

其中

$$\|U_M^N-U_M^{2N}\|_\infty=\|U(T/N,L/M)-U(T/(2N),L/M)\|_\infty,$$
$$\|U_M^N-U_{2M}^N\|_\infty=\|U(T/N,L/M)-U(T/N,L/(2M))\|_\infty,\quad(70)$$

以及时间和空间方向的

$$\text{收敛阶}=\begin{cases}\log_2[E(\tau)/E(\tau/2)],\text{在时间上},\\ \log_2[E(M)/E(2M)],\text{在空间上},\end{cases}\quad(71)$$

定义相对质量误差为

$$RG^n=|(G^n-G^0)/G^0|,\quad(72)$$

相对能量误差为

$$RH^n=|(H^n-H^0)/H^0|,\quad(73)$$

这里 G^n 和 H^n 分别表示在 t_n 处的质量和能量.

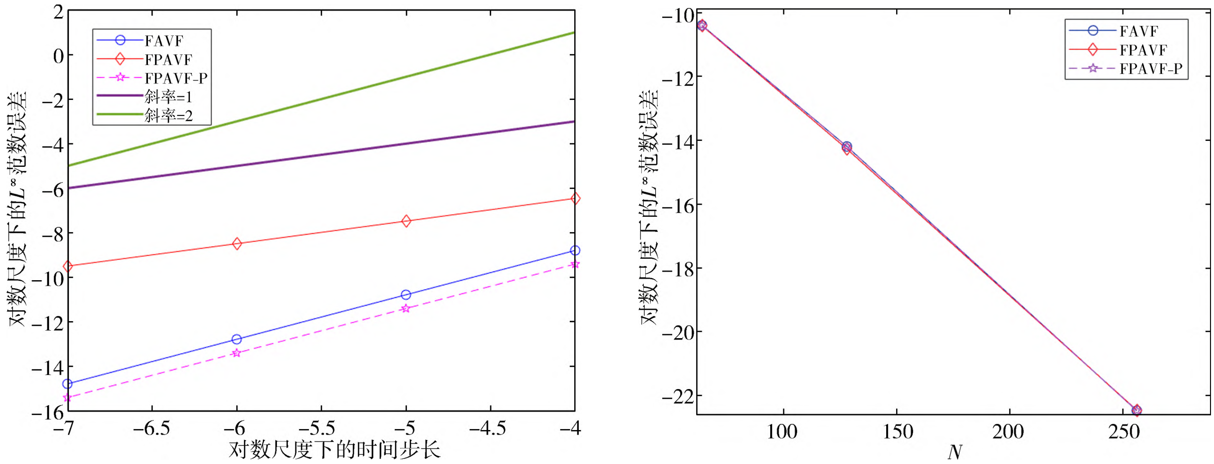
例 4.1 对于二维 SCFSEs(1)和(2),设其初始条件为

$$U(x,0) = \operatorname{sech}(x+10)\exp(3\mathrm{i}x)\operatorname{sech}(y+10)\exp(3\mathrm{i}y),\tag{74}$$

$$V(x,0) = \operatorname{sech}(x-10)\exp(-3\mathrm{i}x)\operatorname{sech}(y-10)\exp(-3\mathrm{i}y),\tag{75}$$

其中 $x=(x,y)$. 在数值算例中,取计算区域为 $(x,y)=[-21,21]\times[-21,21]$, $t\in[0,T]$.

首先测试本文所提 FPAVF 系列方法的准确性. 不失一般性,考虑 $\gamma_1=1,\gamma_2=2,\gamma_3=3,\gamma_4=1,\gamma_5=1$ 时本文所提方法的时间和空间误差. 图 1(a)绘制了 $\alpha=1.7,N=16$ 时 FAVF、FPAVF 和 FPAVF-P 方法的时间误差. 从图 1 可以看出,FAVF 和 FPAVF-P 的斜率为 2,而 FPAVF 的斜率为 1. 这表明 FAVF 方法和 FPAVF-P 方法在时间方向都有 2 阶精度,FPAVF 方法在时间方向只有 1 阶精度. 类似地,图 1(b)绘制了 $\alpha=1.7,\tau=10^{-3}$ 时的空间误差,这表明本文所提方法在空间方向具有谱精度. 这与前面的理论是一致的.

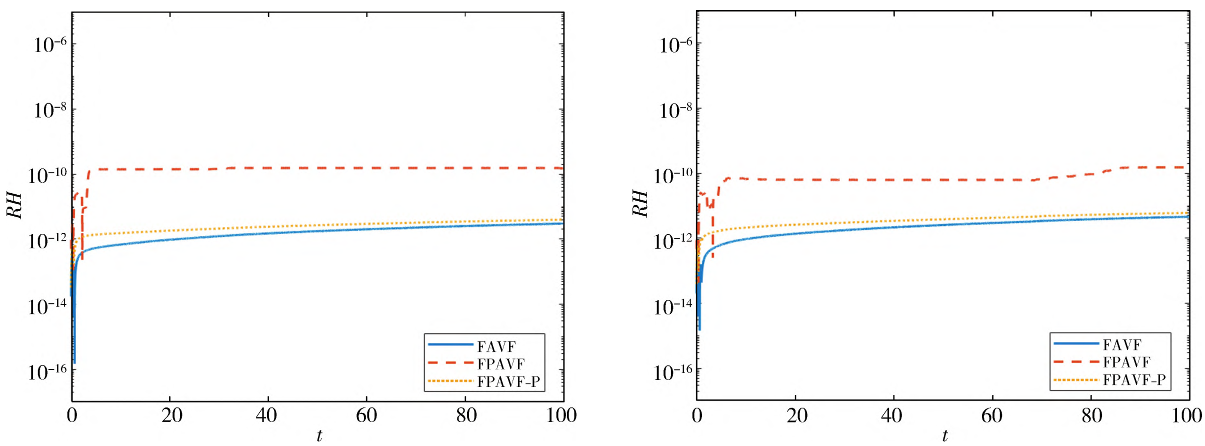


(a) $\alpha=1.7,N=16$ 的时间误差 (b) $\alpha=1.7,\tau=0.001$ 时的空间误差

图 1 3 种方法的时间精度和空间精度

Fig. 1 Convergence orders in time and space of the three methods

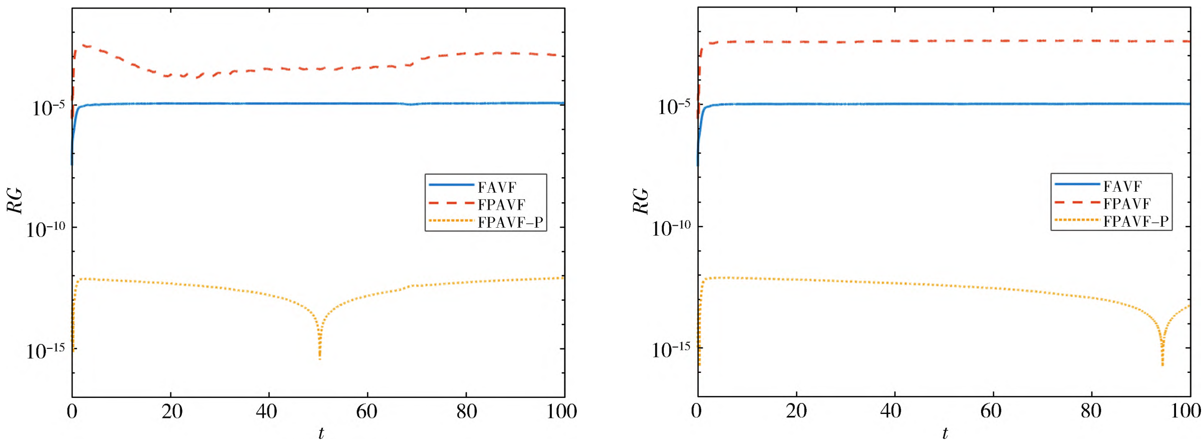
其次验证所提守恒方法的长时间离散守恒性能. 令 $T=100,N=16,\tau=0.01$. 图 2 和 3 分别展示了不同的 α 时离散质量 G^n 和能量 H^n 的相对误差. 可以发现,上述 3 种方法都能很好地保持能量守恒,但只有 FPAVF-P 方法能够呈现质量守恒.



(a) $\alpha=1.4$ (b) $\alpha=2.0$

图 2 不同方法在不同 α 值时的相对能量误差

Fig. 2 Relative errors of energy for different α



(a) $\alpha=1.4$　　(b) $\alpha=2.0$

图 3　不同方法在不同 α 值时的相对质量误差

Fig. 3　Relative errors of mass for different α

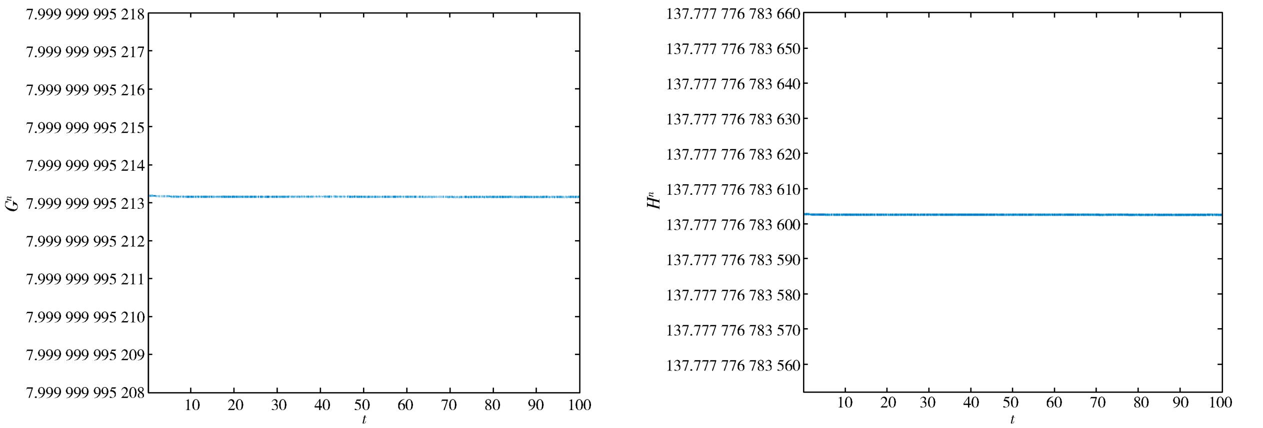
表 1 列出了上述 3 种方法当 $\alpha=2$ 时,在不同时间 t 处计算得到的离散质量和能量. 从表 1 可以发现这 3 种方法都严格保持原始能量,而只有 FPAVF-P 方法保持原始质量.

表 1　$\alpha=2, t=t_n$ 时的离散能量 H^n 和质量 G^n

Tab. 1　Discrete energy H^n and discrete mass G^n at time $t=t_n$ when $\alpha=2$

t	AVF	PAVF	PAVF-P	AVF	PAVF	PAVF-P
0	137.777 776 783 814	137.777 776 783 844	137.777 776 783 814	8.000 193 927 104 56	8.001 136 337 312 08	7.999 999 995 213 18
5	137.777 776 783 812	137.777 776 783 846	137.777 776 783 813	8.000 308 514 210 55	8.001 208 277 009 19	7.999 999 995 213 18
10	137.777 776 783 812	137.777 776 783 852	137.777 776 783 813	8.000 221 341 406 89	8.001 020 624 847 55	7.999 999 995 213 15
20	137.777 776 783 812	137.777 776 783 951	137.777 776 783 813	8.000 229 545 459 51	8.001 210 188 138 15	7.999 999 995 213 15
40	137.777 776 783 812	137.777 776 783 958	137.777 776 783 813	8.000 222 356 026 22	8.001 191 116 478 98	7.999 999 995 213 16
80	137.777 776 783 812	137.777 776 783 893	137.777 776 783 813	8.000 178 020 536 64	8.001 259 704 174 13	7.999 999 995 213 15
100	137.777 776 783 812	137.777 776 783 831	137.777 776 783 813	8.000 162 428 458 12	8.001 186 424 091 21	7.999 999 995 213 15
		原始能量 $E(t_0)$:137.777 777 704 875			原始质量 $Q(t_0)$:7.999 999 995 536 86	

图 4 展示了 $\alpha=2$ 时,由 FPAVF-P 方法计算得到的离散能量和质量随时间的变化图. 从图 4 可以看出,无论时间 t 如何变化,由 FPAVF-P 方法计算得到的离散能量和质量始终不变. 值得说明的是,



(a) FPAVF-P 方法的离散质量 G^n　　(b) FPAVF-P 方法的离散能量 H^n

图 4　$\alpha=2$ 时,FPAVF-P 方法的离散质量 G^n 和离散能量 H^n

Fig. 4　Comparison of prediction results by discrete mass G^n and discrete energy H^n for $\alpha=2$

由于 $\alpha\neq2$ 时的原始能量不易计算,故这里仅展示了 $\alpha=2$ 的结果,但针对其他 α 的值,也可以观察到类似的演化现象.这表明本文所构造的 FPAVF-P 方法确实能够保持方程的原始质量和能量,这验证了前面理论结果的正确性.

4 结论

本文将二维强耦合空间分数阶 Schrödinger 方程重写为一个标准的哈密顿系统,并基于分区平均向量场(PAVF)方法,提出了一种同时质量和能量守恒的守恒方法.并从理论和数值2个角度,验证了该方法的准确性和守恒性质.

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Structure-preserving Method for Strongly Coupled Two-dimensional Fractional Schrodinger Equations

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Abstract: The main contribution of this paper is to construct an effective numerical method for preserving the original invariants of the strongly coupled fractional Schrödinger equations. Firstly, the strongly coupled fractional Schrödinger equations are rewritten into an equivalent Hamiltonian form by using the order reduction technique and the real and imaginary part separation methods. Then, the Fourier pseudo-spectral method and a variety of partitioned average vector field (PAVF) methods are used in the spatial and temporal directions, respectively, and the corresponding fully discrete numerical methods are established. Theoretical and numerical results show that these obtained PAVF methods can preserve the original energy of the studied model, but only the PAVF-P method can preserve the original energy and mass.

Keywords: Hamiltonian system; coupled Schrödinger equation; average vector field method; Fourier pseudo-spectral method

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